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Note on Creasy's Confidence Limits for the Gradient in the Linear Functional Relationship

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An apparent gap in the proof of Creasy's t test is filled and some false statements on this topic, found in the literature, are corrected. Creasy's test is shown to be identical with Williams' t test. The latter can be generalized to the multivariate case of a linear functional relationship.

Let a linear functional relationship $\eta_i = \alpha + \beta\xi_i$, $i = 1,..., n$, be given, the variables of which are measured with errors: $x_i = \xi_i + v_i$, $y_i = \eta_i + w_i$, such that the v_i and w_i are stochastically independent of all the ξ_i and of each other, and v_i and w_i are both distributed as $N(0, \sigma^2)$. No assumptions are made about the variables ξ_i (they may be stochastic or nonstochastic variates).

Since the variances of v_i and w_i are known to be equal, β is identifiable and can be estimated by the principle of orthogonal least squares (Lindley [3]). Call $\hat{\beta}$ the estimator resulting from this principle. Now writing $\beta = \tan\varphi$ and $\hat{\beta} = \tan\hat{\varphi}$, Creasy [1] proves that

$$t = \frac{1}{2}\left[(n-2)\frac{(s_{xx}-s_{yy})^2 + 4s_{xy}^2}{s_{xx}s_{yy} - s_{xy}^2}\right]^{1/2} \sin 2(\hat{\varphi} - \varphi) \sim t_{n-2}. \quad (1)$$

That is, t is distributed as Student's t with $n - 2$ degrees of freedom. She uses this fact to derive confidence limits for φ.

The proof of (1) is somewhat tricky and can easily give rise to misunderstandings. It is the purpose of this note to clarify some of those. Creasy starts out with the special case $\varphi = 0$. In this case y_i, given x_i, is conditionally distributed as $N(\alpha, \sigma^2)$ and therefore

$$t' = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}} \quad (2)$$

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is distributed as Student's t with $n-2$ degrees of freedom, where r is the empirical correlation coefficient of x_i and y_i, $i=1,\dots,n$. With the help of algebraic manipulations (cf. Moran [4]) t' is seen to be equal to the expression in (1) with $\varphi=0$. Now Creasy, in considering the general case $\varphi\neq 0$, says that this can be dealt with by rotating to new axes, and from this (1) is obtained.

The apparent gap in the proof has puzzled some writers. Kendall and Stuart [2, p. 414] try to fill the gap by asserting that the "variances and covariances are invariant under orthogonal transformations." But this is certainly not true. Moran [4] believes that (1) is incorrect for $\varphi\neq 0$ and that, in order to test a hypothesis $\varphi=\varphi_0$, a rotation of axes by the angles φ_0 has in fact to be carried out. A closer examination of (1) reveals, however, that rotating the axes is not necessary in practice, though it *is* needed in order to prove (1) for general φ starting from the special case $\varphi=0$.

Consider the general case $\varphi\neq 0$ and rotate the axes by the angle φ. This is equivalent to transforming $z_i=(x_i,y_i)$ to new variates $z_i^*=(x_i^*,y_i^*)$ by an orthogonal transformation P: $z_i^*=z_iP$. (ξ_i,η_i) and (v_i,w_i) are transformed in the same way. The new variates obey a model of the same kind as the old ones with the only difference that the new slope of the linear relationship is zero. Therefore (1) is applicable with (x^*,y^*) and $\hat\varphi^*$ in place of (x,y) and $\hat\varphi$, and with $\varphi=0$; i.e.,

$$t^*=\frac{1}{2}\left[(n-2)\frac{(s_{x^*x^*}-s_{y^*y^*})^2+4s_{x^*y^*}}{s_{x^*x^*}s_{y^*y^*}-s_{x^*y^*}^2}\right]^{1/2}\sin 2\hat\varphi^*\sim t_{n-2}. \quad (3)$$

Now let

$$S=\begin{pmatrix} s_{xx} & s_{xy}\\ s_{yx} & s_{yy}\end{pmatrix},\qquad S^*=\begin{pmatrix} s_{x^*x^*} & s_{x^*y^*}\\ s_{y^*x^*} & s_{y^*y^*}\end{pmatrix};$$

then $S^*=P'SP$. Call A and A^* the expression under the square root in (1) and (3), respectively. Then it can be shown that $A=A^*$. Indeed A can be expressed as a function of the trace and the determinant of S:

$$A=(n-2)\frac{(\operatorname{tr} S)^2-4\det S}{\det S}.$$

An analogous expression holds for A^*. But $\operatorname{tr} S$ and $\det S$ are invariant under orthogonal transformations, i.e., $\operatorname{tr} S^*=\operatorname{tr} S$, $\det S^*=\det S$, and therefore $A=A^*$. By noting that $\hat\varphi^*=\hat\varphi-\varphi$ (because $\hat\varphi^*$, just as $\hat\varphi$, is found by the principle of orthogonal least squares, a principle which is invariant under rotation of axes) it can now be seen that $t^*=t$, and this proves the assertion of (1) for general φ.

In concluding I should like to draw attention to a different approach followed by Williams [6].[1] He starts out with the general case $\beta \neq 0$ right from the beginning and considers the variates $u_i = y_i - \beta x_i$ and $z_i = x_i + \beta y_i$. It can be shown that u_i, given z_i, is normally distributed with constant mean and variance. Therefore a similar statistic as (2) can be defined, but this time with r being the empirical correlation coefficient of u_i and z_i, $i = 1,..., n$. It is again distributed as t_{n-2}. From this, Williams derives the following result:

$$\tau = \sqrt{n-2}\,\frac{\beta^2 s_{xy} + \beta(s_{xx} - s_{yy}) - s_{xy}}{(1+\beta^2)\sqrt{s_{xx}s_{yy} - s_{xy}^2}} \sim t_{n-2}.$$

Now it can be shown by some algebraic manipulations that $\tau = t$ as defined by (1), and thus τ gives rise to the same confidence limits for β as t.

Williams' approach can easily be generalized to the case of a multiple linear relationship. Suppose the following relationship is given (where the observational index $i = 1,..., n$ is suppressed for simplicity): $\eta = \alpha + \beta_1 \xi_1 + \beta_2 \xi_2$ with $y = \eta + w$, $x_j = \xi_j + v_j$, $j = 1, 2$. Assume w to be independent of (v_1, v_2) and for simplicity also assume v_1 to be independent of v_2. The variances σ_j^2 of v_j and σ_w^2 of w need not be equal but are assumed to be known up to a common factor. All the other assumptions of the simple model should be carried over into the multiple model. Define $u = y - \beta_1 x_1 - \beta_2 x_2$ and $z_j = x_j/\sigma_j^2 + \beta_j y/\sigma_w^2$, $j = 1, 2$. Then u, given (z_1, z_2), is normally distributed with constant mean and variance. Therefore

$$F = \frac{(n-3)R^2}{2(1-R^2)}$$

is distributed as F_{n-3}^2, where R is the empirical coefficient of multiple correlation of u with respect to (z_1, z_2) computed from a sample of these variables.

The statistic F can be used to test the hypothesis $(\beta_1, \beta_2) = (\beta_1^0, \beta_2^0)$ or to construct a confidence region for (β_1, β_2).

References

[1] Creasy, M. A. (1956). Confidence limits for the gradient in the linear functional relationship. *J. Roy. Stat. Soc. B* **18**, 65–69.

[2] Kendall, M. G., and Stuart, A. (1979). *The Advanced Theory of Statistics.* Vol. 2: *Inference and Relationship,* 4th ed. Hafner, New York/London.

[3] Lindley, D. V. (1947). Regression lines and the linear functional relationship. *J. Roy. Stat. Soc. B* **9**, 218–244.

[1] See also Williams [5, p. 199].

[4] MORAN, P. A. P. (1971). Estimating structural and functional relationships. *J. Multivar. Anal.* **1**, 232–255.

[5] WILLIAMS, E. J. (1959). *Regression Analysis.* Wiley, New York/London.

[6] WILLIAMS, E. J. (1973). Tests of correlation in multivariate analysis. *Bull. Int. Stat. Inst. Proc. 39th Session*, **45**, Book 4, 218–232.